Lecture 2: Representations of Accelerator Elements
maps, kicks and beam propagation

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0.1 Lagrangian and Hamiltonian Dynamics

The motion of charged particles in electromagnetic fields could be described through the Lorentz force equations. For a particle of charge q and mass m, they are given in MKSA units by

$$\frac{d\vec{p}^{\rm mech}}{dt} = q\vec{E} + q\vec{v} \times \vec{B}, \tag{1}$$

where $\vec{v}$ is the particle's velocity, and where $\vec{E}$ and $\vec{B}$ are the electric and magnetic field, respectively. The quantity $\vec{p}^{\rm mech}$ denotes the mechanical momentum, which is given in Cartesian coordinates by

$$\vec{p}^{\rm mech} = \gamma m\vec{v}, \tag{2}$$

where $\gamma = \sqrt{1-\beta^2}$, $\beta^2 = \vec{\beta}\cdot\vec{\beta}$, and $\vec{\beta} = \vec{v}/c$. The dynamics described by the Lorentz force equation can be reformulated in terms of Lagrange's equations,

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0, \tag{3}$$

where $L(\vec{q}, \dot{\vec{q}}_i, t)$ denotes the Lagrangian, $\vec{q} = (q_1, q_2, \ldots)$ denotes a set of generalized coordinates, and where a dot denotes d/dt. For a charged particle in electromagnetic fields,

$$L = -mc^2\sqrt{1-\beta^2} - q\psi + q\vec{v}\cdot\vec{A}, \tag{4}$$

where $\psi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ are the scalar and vector potentials, respectively. In MKSA units, these are related to the electric and magnetic field according to

$$\vec{B} = \nabla \times \vec{A}, \tag{5}$$

$$\vec{E} = -\nabla\psi - \frac{\partial \vec{A}}{\partial t}. \tag{6}$$

The canonical momentum, p_i, which is conjugate to the generalized coordinate, q_i, is

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}. \tag{7}$$

The Hamiltonian $H(\vec{q}, \vec{p}, t)$ is related to the Lagrangian, $L(\vec{q}, \dot{\vec{q}}, t)$ according to

$$H(\vec{q}, \vec{p}, t) = \sum_i p_i\dot{q}_i - L, \tag{8}$$

where Eq. (7) is used to reexpress $\dot{q}_i$ in terms of p_i. Hamilton's equations are given by

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{9}$$

It follows from Eq. (7) and Eq. (4) that, in Cartesian coordinates, the canonical momenta are given by

$$\vec{p} = \gamma m\vec{v} + q\vec{A} = \vec{p}^{\rm mech} + q\vec{A}. \tag{10}$$

(In these notes we will use the convention that, unless it is specifically stated or annotated otherwise, a quantity p denotes a *canonical* momentum.) It follows that the Hamiltonian for a particle of mass m and charge q, in Cartesian coordinates, is given by

$$H = \left[m^2c^4 + c^2(\vec{p} - q\vec{A})^2\right]^{1/2} + q\psi, \tag{11}$$

which is just the total energy, $\sqrt{(p^{\rm mech})^2c^2 + m^2c^4} + q\psi$.

A key concept in our approach to analyzing charged particle dynamics in accelerators and beam transport systems is the *transfer map*. Let $H(\zeta, t)$ denote the Hamiltonian of some dynamical system, where $\zeta = (q_1, p_1, q_2, p_2, \ldots, q_m, p_m)$. The $2m$-dimensional space whose axes are $q_1, p_1, q_2, p_2, \ldots, q_m, p_m$ is called *phase space*. The $2m+1$ dimensional space that is the direct product of phase space with the time axis is called *state space*. As a particle evolves in time, its trajectory traces out a path in state space (and also in phase space). The set of all such trajectories in state space for all possible initial conditions is called a *Hamiltonian flow*. If we choose some initial time, t^{in}, and some final time, $t^{\rm fin}$, then Hamilton's equations, or equivalently the Hamiltonian flow, may be regarded as defining a generally nonlinear mapping, $\mathcal{M}$, that maps $\zeta(t^{\rm in})$ into $\zeta(t^{\rm fin})$. We will use the notation $\zeta^{\rm in} = \zeta(t^{\rm in})$ and $\zeta^{\rm fin} = \zeta(t^{\rm fin})$, and we will write

$$\zeta^{\rm fin} = \mathcal{M}\zeta^{\rm in}. \tag{12}$$

The quantity $\mathcal{M}$ is called the *transfer map* relating $\zeta^{\rm in}$ and $\zeta^{\rm fin}$. It will turn out that, because $\mathcal{M}$ comes from a Hamiltonian, it belongs to the class of mappings known as *symplectic* mappings.

0.1.1 Another take on Symplectic Integration

Symplectic integration algorithms are numerical algorithms that preserve the underlying symplectic nature of the equations to be integrated. The first 3rd-order symplectic integrator was developed by Ronald Ruth [2]; it was applicable to Hamiltonian systems of a particular form, namely $H = A(\vec{q}) + V(\vec{p})$. Later, Etienne Forest, Ruth, Fillipo Neri, and others, went on to develop more general 4th-order symplectic algorithms based on Lie methods [3]. Finally, Yoshida showed how, given an algorithm of order $2n$ (subject to certain conditions), one can generate an algorithm of order $2n+2$ [4]. The wide applicability of Yoshida's result was not fully appreciated until it was identified and exploited by Forest et al. [5].

There is not an *explicit* symplectic integration algorithm that works for arbitrary Hamiltonians. However, there are explicit methods for Hamiltonians that can be written as a sum of terms, each of which can be solved separately. Since any monomial of the phase space variables, $(x^i p_x^j y^k p_y^l)$ is itself integrable, it follows that any Hamiltonian that is a finite sum of monomials (as often occurs in magnetic optics) can be treated using these so-called *split-operator symplectic integration* techniques. In principle these can be extended to any order using the method of Yoshida. There is also a 2nd order *implicit* algorithm that can be applied to *any* Hamiltonian, which can also be extended to high order using the method of Yoshida. All of these are discussed below.

0.1.2 Split-Operator Symplectic Integration

Consider a Hamiltonian that can be written as a sum of two terms,

$$H = H_1 + H_2. \tag{13}$$

Suppose we can obtain the mapping $\mathcal{M}_1$ corresponding to H_1 and the mapping $\mathcal{M}_2$ corresponding to H_2. Then the following algorithm is accurate through 2nd order:

$$\mathcal{M}(\tau) = \mathcal{M}_1(\tau/2)\ \mathcal{M}_2(\tau)\ \mathcal{M}_1(\tau/2). \tag{14}$$

This approach is easily generalized to more splittings. For example, consider a Hamiltonian that can be written as a sum of three terms,

$$H = H_1 + H_2 + H_3. \tag{15}$$

Then a 2nd order algorithm is given by

$$\mathcal{M}(\tau) = \mathcal{M}_1(\tau/2)\ \mathcal{M}_2(\tau/2)\ \mathcal{M}_3(\tau)\ \mathcal{M}_2(\tau/2)\ \mathcal{M}_1(\tau/2). \tag{16}$$

Returning to the two-term case, Forest and Ruth showed that the following algorithm is accurate through 4th order:

$$\mathcal{M}(\tau) = \mathcal{M}_1(\frac{s}{2})\ \mathcal{M}_2(s)\ \mathcal{M}_1(\frac{\alpha s}{2})\ \mathcal{M}_2((\alpha-1)s)\ \mathcal{M}_1(\frac{\alpha s}{2})\ \mathcal{M}_2(s)\ \mathcal{M}_1(\frac{s}{2}), \tag{17}$$

where

$$\alpha = 1 - 2^{1/3}, \quad s = \tau/(1+\alpha). \tag{18}$$

In other words, this is a 4th-order algorithm that is a product of seven maps. Note that α is approximately equal to -0.26, and in particular α is negative. This means that this middle three steps in the seven-step procedure have negative time steps.

0.1.3 The Method of Yoshida

The main result due to Yoshida, as well as the wide applicability of Yoshida's approach, is described in Ref. [4]. Let $\mathcal{M}_{2n}$ denote a mapping that is an approximate solution to a problem that is accurate through order $2n$. Also, suppose $\mathcal{M}_{2n}$ has the property that

$$\mathcal{M}_{2n}(\tau)\mathcal{M}_{2n}(-\tau) = I, \tag{19}$$

where I is the identity mapping. Then the following is accurate through order $2n+2$:

$$\mathcal{M}_{2n+2}(\tau) = \mathcal{M}_{2n}(z_0\tau)\ \mathcal{M}_{2n}(z_1\tau)\ \mathcal{M}_{2n}(z_0\tau) \tag{20}$$

where

$$z_0 = \frac{1}{2-2^{1/(2n+1)}}, \qquad z_1 = \frac{-2^{1/(2n+1)}}{2-2^{1/(2n+1)}} \tag{21}$$

- **Problem**
  Consider the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{\alpha^2}{2}(x^2+y^2). \tag{22}$$

It can be split as $H = H_1 + H_2$, where

$$H_1 = \frac{1}{2}(p_x^2 + p_y^2), \qquad H_2 = \frac{\alpha^2}{2}(x^2+y^2). \tag{23}$$

Show that the map $\mathcal{M}_1$ for H_1 is given by

$$x^{fin} = x^{in} + p_x^{in}\tau, \qquad p_x^{fin} = p_x^{in},$$
$$y^{fin} = y^{in} + p_y^{in}\tau, \qquad p_y^{fin} = p_y^{in}. \tag{24}$$

an the map $\mathcal{M}_2$ for H_2 is given by

$$x^{fin} = x^{in}, \qquad p_x^{fin} = p_x^{in} - \alpha^2 x^{in}\tau,$$
$$y^{fin} = y^{in}, \qquad p_y^{fin} = p_y^{in} - \alpha^2 y^{in}\tau. \tag{25}$$

Implement the 2nd order symplectic integrator and plot phase-space trajectories for your choice of parameters.

0.1.4 Coordinate as an Independent Variable

Accelerator elements produce fields that are localized in their vicinity. Therefore, in analogy with optics, it is useful to treat magnets as self-contained objects, or lenses. To study a beam transport system made of such elements it is useful to choose a coordinate as the independent variable, rather than the time. For example, in systems where the design orbit is a straight line (call it the z-axis), one could choose the coordinate z as the independent variable. *To appreciate the necessity of such a choice, think of the actual tracking implementation through a thick element, with time as the independent variable: for every time step you hage to ask "am I out yet?"; not very efficient.* To do this, define a new variable, p_t, by

$$p_t = -H(\vec{q}, \vec{p}, t). \tag{26}$$

In other words, p_t is just the negative of the single particle total energy. Next, invert the above equation to obtain p_z as a function of (x, p_x, y, p_y, t, p_t) and z. By the Implicit Function Theorem, we can do this so long as $\partial H/\partial p_z \neq 0$ in the region of interest (i.e. so long as $\dot{z}$ is nonzero). Lastly, define a quantity K according to

$$K(x, p_x, y, p_y, t, p_t; z) = -p_z. \tag{27}$$

Then it is easy to show (by the chain rule) that

$$x' = \frac{\partial K}{\partial p_x}, \qquad p_x' = -\frac{\partial K}{\partial x}, \tag{28}$$
$$y' = \frac{\partial K}{\partial p_y}, \qquad p_y' = -\frac{\partial K}{\partial y}, \tag{29}$$
$$t' = \frac{\partial K}{\partial p_t}, \qquad p_t' = -\frac{\partial K}{\partial t}, \tag{30}$$

where a prime denotes d/dz. In other words, K is the new Hamiltonian with z as the independent variable. The new Hamiltonian is given by

$$K(x, p_x, y, p_y, t, p_t; z) = -\left[(p_t + q\psi)^2/c^2 - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2\right]^{1/2} - qA_z. \tag{31}$$

Similarly, if we use θ as the independent variable in cylindrical coordinates, it follows that

$$K(r, p_r, z, p_z, t, p_t; \theta) = -r\left[(p_t + q\psi)^2/c^2 - m^2c^2 - (p_r - qA_r)^2 - (p_z - qA_z)^2\right]^{1/2} - qrA_\theta. \tag{32}$$

Most of our examples that follow will utilize z as the independent variable. However, when discussing formalism one often uses the symbol t in a generic sense to denote the independent variable. The reader should keep in mind that, when we say "time-dependent" we often mean "z-dependent," and references to t^{in} and t^{fin} (initial and final times) often mean z^{in} and z^{fin}.

0.1.5 Canonical Transformations

In what follows it will prove useful to perform various canonical transformations from one set of variables, $(\vec{q}, \vec{p})$, to new variables, $(\vec{Q}, \vec{P})$. Concurrently, the original Hamiltonian, $H(\vec{q}, \vec{p})$, will be transformed into a new Hamiltonian $H^{\rm new}(\vec{Q}, \vec{P})$. For a general discussion, see, e.g., Goldstein [1]. For our purposes, two types of transforms will be used frequently.

First consider a scaling transformation, for which

$$Q_i = q_i/a_i, \quad P_i = p_i/b_i \qquad (i = 1, 2, 3), \tag{33}$$

where a_i and b_i are constants. It is easy to show that, so long as the product $a_i b_i \equiv ab$ is the same for all i, the transformation is canonical, and the new Hamiltonian is given by

$$H^{\rm new}(\vec{Q}, \vec{P}) = \frac{1}{ab} H(a\vec{Q}, b\vec{P}). \tag{34}$$

Next consider a canonical transformation based on the following generating function,

$$F_2(\vec{q}, \vec{P}, t) = (x - x^g)(P_x + p_x^g) + (y - y^g)(P_y + p_y^g) + (z - z^g)(P_z + p_z^g), \tag{35}$$

where $x^g, p_x^g, y^g, p_y^g, z^g, p_z^g$ are functions of t. Based on this generating function, the new variables are related to the old variables according to

$$\vec{Q} = \frac{\partial F_2}{\partial \vec{P}}, \qquad \vec{p} = \frac{\partial F_2}{\partial \vec{q}}, \tag{36}$$

and the new Hamiltonian is given by

$$H^{\rm new}(\vec{Q}, \vec{P}) = H + \frac{\partial F_2}{\partial t}. \tag{37}$$

In this case, we obtain

$$\begin{aligned} X = x - x^g, \qquad P_x &= p_x - p_x^g, \\ Y = y - y^g, \qquad P_y &= p_y - p_y^g, \\ Z = z - z^g, \qquad P_z &= p_z - p_z^g. \end{aligned} \tag{38}$$

- **Problem 1.4**
  Show that Eqs. (35) and (36) lead to (38).

Lastly, suppose that $(x^g, p_x^g, y^g, p_y^g, z^g, p_z^g)$ denotes a particular solution of Hamilton's Equations with Hamiltonian H, which we will call a reference trajectory. Then, from Eq. (38), the new variables are deviations from the reference trajectory. Furthermore, one can show that the new Hamiltonian, $H^{\rm new}(\vec{Q}, \vec{P}, t)$, if expanded in the new variables, will contain no linear terms.

0.2 Equations for the Linear Map

In this section we will deal with the analytical and numerical computation of linear transfer maps. Transfer map techniques provide a useful framework in which to analyze and optimize beam transport systems. By analyzing and manipulating maps, instead of simply tracking single particle trajectories, one can design transport systems to have desired properties, such as the absence or minimization of certain nonlinear effects. Analytical representations of transfer maps are available for only a few idealized beamline elements, and, in particular, analytical results are not available for beamline elements with realistic fringe fields. As a result, we must often resort to computing maps numerically rather than using analytical formulas.

In accelerator physics one is usually interested in trajectories near some particular trajectory, referred to as the "design," "reference," "synchronous", or "fiducial" trajectory. We will usually call this as the "given" trajectory, denoted by a superscript "g", as in ζ^g. Occasionally we will also use the symbol "o," as in p^o, β_o, and γ_o.

We will start by dealing with systems for which it is natural to use z as the independent variable. As stated previously, the Hamiltonian for such a system is obtained by setting the original Hamiltonian, $H(x, p_x, y, p_y, z, p_z; t)$, equal to $-p_t$, and solving for $-p_z$. As shown in Eq. (31), the new Hamiltonian, $K(x, p_x, y, p_y, t, p_t; z)$, is

$$K = -\left[(p_t + q\psi)^2/c^2 - m^2c^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2\right]^{1/2} - qA_z. \tag{39}$$

In all that follows it will be convenient to work with dimensionless variables. Let

$$\bar{x} = x/l, \qquad \bar{p}_x = p_x/\delta \tag{40}$$
$$\bar{y} = y/l, \qquad \bar{p}_y = p_y/\delta \tag{41}$$
$$\bar{t} = \omega t, \qquad \bar{p}_t = p_t/(\omega l \delta) \tag{42}$$

where l, δ and ω are constants. The quantity l is a scale length, δ is a scale momentum, and ω is a scale frequency (i.e. $1/\omega$ is a scale time). For most problems we will set $l = 1$ m. The choices of δ and ω will depend on the situation. When treating magnetostatic elements, it is conventional to set δ equal to the design momentum, p^o, of the system under consideration. When dealing with acceleration by rf cavities, we cannot scale according to p^o, since the design momentum is not a constant; instead we set $\delta = mc$. In this case it is also useful to set ω equal to the frequency of the rf fields, or some multiple thereof. The transformation to dimensionless variables is canonical, and the new variables are governed by the following Hamiltonian, K^{new}:

$$\begin{aligned} &K^{\text{new}}(\bar{x}, \bar{p}_x, \bar{y}, \bar{p}_y, \bar{t}, \bar{p}_t) = \\ &-\tfrac{1}{l}\left[(\tfrac{\omega l}{c}\bar{p}_t + \tfrac{q}{\delta c}\psi)^2 - (\tfrac{mc}{\delta})^2 - (\bar{p}_x - \tfrac{q}{\delta}A_x)^2 - (\bar{p}_y - \tfrac{q}{\delta}A_y)^2\right]^{1/2} - \tfrac{q}{\delta l}A_z, \end{aligned} \tag{43}$$

where

$$\vec{A} = \vec{A}(l\bar{x}, l\bar{y}, \bar{t}/\omega; z) \tag{44}$$
$$\psi = \psi(l\bar{x}, l\bar{y}, \bar{t}/\omega; z). \tag{45}$$

Lastly, go to the reference trajectory by setting

$$X = \bar{x} - \bar{x}^g, \qquad P_x = \bar{p}_x - \bar{p}_x^g \tag{46}$$
$$Y = \bar{y} - \bar{y}^g, \qquad P_y = \bar{p}_y - \bar{p}_y^g \tag{47}$$
$$T = \bar{t} - \bar{t}^g, \qquad P_t = \bar{p}_t - \bar{p}_t^g \tag{48}$$

This transformation is also canonical, so the deviation variables are also governed by a Hamiltonian, which we will denote H. The new Hamiltonian is

$$\begin{aligned} &H(X, P_x, Y, P_y, T, P_t) = -\tfrac{q}{\delta l}A_z \\ &-\tfrac{1}{l}\left[(\tfrac{\omega l}{c}P_t + \tfrac{\omega l}{c}\bar{p}_t^g + \tfrac{q}{\delta c}\psi)^2 - (\tfrac{mc}{\delta})^2 - (P_x + \bar{p}_x^g - \tfrac{q}{\delta}A_x)^2 - (P_y + \bar{p}_y^g - \tfrac{q}{\delta}A_y)^2\right]^{1/2} \\ &-\tfrac{d\bar{x}^g}{dz}(P_x + \bar{p}_x^g) + \tfrac{d\bar{p}_x^g}{dz}X - \tfrac{d\bar{y}^g}{dz}(P_y + \bar{p}_y^g) + \tfrac{d\bar{p}_y^g}{dz}Y - \tfrac{d\bar{t}^g}{dz}(P_t + \bar{p}_t^g) + \tfrac{d\bar{p}_t^g}{dz}T \end{aligned} \tag{49}$$

where

$$\vec{A} = \vec{A}(lX + l\bar{x}^g, lY + l\bar{y}^g, (T + \bar{t}^g)/\omega; z) \tag{50}$$
$$\psi = \psi(lX + l\bar{x}^g, lY + l\bar{y}^g, (T + \bar{t}^g)/\omega; z). \tag{51}$$

(Often many terms in Eq. (49) will be zero, as would be the case, for example, if $x^g = p_x^g = y_g = p_y^g = 0$). At this stage we expand H around the given trajectory, $X = P_x = Y = P_y = T = P_t = 0$. By construction it will turn out that all the linear terms vanish, as demostrated in Problem 1.5. Also, we can drop any term that is a constant or just a function of z, since such terms do not affect the resulting equations of motion. Thus, we obtain

$$H = H_2 + H_3 + H_4 + \cdots, \tag{52}$$

where each H_n is a homogeneous polynomial of degree n in (X, P_x, Y, P_y, T, P_t). The linear dynamics are governed by H_2, and the quantities H_2 through H_n determine the dynamics through order $n-1$.

It can be shown that the linear transfer map, M, obeys the differential equation

$$\frac{dM}{dt} = JSM, \tag{53}$$

where S is a symmetric matrix defined in terms of H_2 according to

$$H_2 = \frac{1}{2}\sum_{a,b=1}^{2m} S_{ab}\zeta_a\zeta_b. \tag{54}$$

and where ζ denotes the collection of canonical coordinates and momenta. For example, in one dimension, with $\zeta = (x, p_x)$, if $H = \frac{1}{2}ax^2 + bxp_x + \frac{1}{2}cp_x^2$, then

$$S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{55}$$

In Eq. (53), the precise form of the matrix J depends on the ordering of the variables in the definition of ζ. For example, if we set $\zeta = (X, P_x, Y, P_y, T, P_t)$, then J is a matrix that has copies of a 2x2 matrix J_2 on the diagonal and zeros elsewhere, with

$$J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{56}$$

So, for example, in three dimensions,

$$J = \left(\begin{array}{cc|cc|cc} 0 & 1 & & & & \\ -1 & 0 & & & & \\ \hline & & 0 & 1 & & \\ & & -1 & 0 & & \\ \hline & & & & 0 & 1 \\ & & & & -1 & 0 \end{array}\right). \tag{57}$$

If, instead, we had set $\zeta = (X, Y, T, P_x, P_y, P_t)$, then J would be of the form

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{58}$$

where I is the 3x3 identity matrix.

0.3 Rectilinear Magnetostatic Elements

For magnetic multipoles excluding bending magnets (i.e. for magnetic quadrupoles, sextupoles, octupoles, etc.), as well as the drift space, we will assume that the design trajectory is along the z-axis of a Cartesian coordinate system:

$$\bar{x}^g = \bar{p}_x^g = \bar{y}^g = \bar{p}_y^g = 0. \tag{59}$$

The temporal variables satisfy

$$t^g = \frac{z}{v^o}, \tag{60}$$
$$p_t^g = constant, \tag{61}$$

where $v^o \equiv v^g$ is the velocity on the design trajectory, $\beta^o = v^o/c$, and $\gamma^o = 1/\sqrt{1-(\beta^o)^2}$. For these elements we will choose

$$\delta = p^o, \tag{62}$$
$$\omega l/c = 1 \ \ (\Rightarrow \omega = c/l), \tag{63}$$
$$l = 1 \text{ m}, \tag{64}$$

where p^o is the design momentum. Recall that p_t is the negative of the total energy, $p_t = -(\gamma mc^2 + q\psi)$. Assuming ψ equals zero on the reference trajectory (in fact, it can be chosen to be identically zero for magnetostatic elements), it follows that $p_t^g = -\gamma mc^2$. But in dimensionless variables, this quantity is scaled by $\omega l\delta$ (see Eq. (42)). This leads to the surprising result that, though p_t^g is proportional to $-\gamma^o$, the dimensionless variable $\bar{p}_t^g$ is given by

$$\bar{p}_t^g = -\frac{1}{\beta^o}. \tag{65}$$

In what follows (see, for example, Eq. (67), we will expand a square root that contains the terms $(p_t^g)^2 - 1/(\gamma^o\beta^o)^2$. Thanks to Eq. (65), this equals one:

$$(p_t^g)^2 - \frac{1}{(\gamma^o\beta^o)^2} = 1. \tag{66}$$

- **Problem**
  Verify Eq. (65) and Eq. (66).

Note Well: From now on we will cease using the notation (X, P_x, Y, P_y, T, P_t), and instead use (x, p_x, y, p_y, t, p_t) to denote **dimensionless deviations from the given trajectory**.

0.3.1 Drift Space

In this case the Hamiltonian given in Eq. (49) reduces to

$$H = -\sqrt{(p_t - \frac{1}{\beta_o})^2 - \frac{1}{(\gamma^o\beta^o)^2} - p_x^2 - p_y^2} - \frac{1}{\beta^o}p_t. \tag{67}$$

Expanding the Hamiltonian, we obtain $H = H_2 + H_3 + H_4 + \ldots$, where

$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2\gamma_o^2\beta_o^2}p_t^2 \tag{68}$$

$$H_3 = -\frac{1}{2\beta_o}(p_x^2+p_y^2)p_t + \frac{1}{2\gamma_o^2\beta_o^2}p_t^3 \tag{69}$$

$$H_4 = \frac{1}{8}(p_x^2+p_y^2)^2 + \frac{1}{4}(\frac{3}{\beta_o^2}-1)(p_x^2+p_y^2)p_t^2 + \frac{1}{8\gamma_o^2\beta_o^2}(\frac{5}{\beta_o^2}-1)p_t^4 \tag{70}$$

The mapping corresponding to H_2 is obviously

$$\begin{aligned} x^{fin} &= x^{in} + p_x^{in} z, & p_x^{fin} &= p_x^{in}, \\ y^{fin} &= y^{in} + p_y^{in} z, & p_y^{fin} &= p_y^{in}, \\ t^{fin} &= t^{in} + \frac{p_t^{in}}{\gamma_o^2\beta_o^2} z, & p_t^{fin} &= p_t^{in}. \end{aligned} \tag{71}$$

In matrix notation, the linear map is given by

$$M = \left(\begin{array}{cc|cc|cc} 1 & z & & & & \\ 0 & 1 & & & & \\ \hline & & 1 & z & & \\ & & 0 & 1 & & \\ \hline & & & & 1 & \frac{1}{\gamma_o^2\beta_o^2}z \\ & & & & 0 & 1 \end{array}\right). \tag{72}$$

Before leaving the treatment of the drift space, we emphasize that, due to the presence of the square root in Eq. (67), the drift is a *nonlinear* element. We can compute the exact, nonlinear map corresponding to Eq. (67) by solving Hamilton's equations. This is easy, since the Hamiltonian contains only momenta and not coordinates. The result is:

$$\begin{aligned} x^{fin} &= x^{in} + \frac{p_x^{in} z}{\sqrt{(p_t^{in}-\frac{1}{\beta_o})^2 - \frac{1}{(\gamma^o\beta^o)^2} - (p_x^{in})^2 - (p_y^{in})^2}}, \\ p_x^{fin} &= p_x^{in}, \\ y^{fin} &= y^{in} + \frac{p_y^{in} z}{\sqrt{(p_t^{in}-\frac{1}{\beta_o})^2 - \frac{1}{(\gamma^o\beta^o)^2} - (p_x^{in})^2 - (p_y^{in})^2}}, \\ p_y^{fin} &= p_y^{in}, \\ t^{fin} &= t^{in} - z\left[\frac{1}{\beta_o} + \frac{(p_t^{in}-1/\beta_o)}{\sqrt{(p_t^{in}-\frac{1}{\beta_o})^2 - \frac{1}{(\gamma^o\beta^o)^2} - (p_x^{in})^2 - (p_y^{in})^2}}\right], \\ p_t^{fin} &= p_t^{in}. \end{aligned} \tag{73}$$

- **Problem**
  Verify that Eq. (73) is the solution of Hamilton's equations with Hamiltonian (67).

0.3.2 Magnetic Quadrupole

Consider a magnetic quadrupole oriented along the z-axis of a Cartesian coordinate system. In this case, the vector and scalar potentials can be represented by

$$\begin{aligned} A_x(x,y,z) &= \frac{1}{4}g'(z)(x^3-xy^2)+\cdots, \\ A_y(x,y,z) &= \frac{1}{4}g'(z)(x^2y-y^3)+\cdots, \\ A_z(x,y,z) &= \frac{1}{2}g(z)(y^2-x^2) - \frac{1}{12}g''(z)(y^4-x^4)+\cdots, \end{aligned} \tag{74}$$

$$\psi = 0, \tag{75}$$

where $g(z)$ denotes the quadrupole gradient (in Tesla/meter, for example), and where a prime denotes d/dz. Taking the curl of $\vec{A}$ gives

$$\begin{aligned} B_x(x,y,z) &= gy - \frac{1}{12}g''(y^3+3x^2y)+\cdots, \\ B_y(x,y,z) &= gx - \frac{1}{12}g''(x^3+3y^2x)+\cdots, \\ B_z(x,y,z) &= g'xy+\cdots \end{aligned} \tag{76}$$

Expanding the Hamiltonian, we obtain $H = H_2 + H_3 + H_4 + \ldots$, where

$$H_2 = \frac{1}{2}(p_x^2+p_y^2) + \frac{1}{2}k(z)(x^2-y^2) + \frac{1}{2\gamma_o^2\beta_o^2}p_t^2 \tag{77}$$

$$H_3 = -\frac{1}{2\beta_o}(p_x^2+p_y^2)p_t + \frac{1}{2\gamma_o^2\beta_o^2}p_t^3 \tag{78}$$

$$\begin{aligned} H_4 = {} & \tfrac{1}{12}k''(y^4-x^4) - \tfrac{1}{4}k'\left[(x^3-xy^2)p_x + (x^2y-y^3)p_y\right] + \tfrac{1}{8}(p_x^2+p_y^2)^2 \\ & + \tfrac{1}{4}(\tfrac{3}{\beta_o^2}-1)(p_x^2+p_y^2)p_t^2 + \tfrac{1}{8\gamma_o^2\beta_o^2}(\tfrac{5}{\beta_o^2}-1)p_t^4 \end{aligned} \tag{79}$$

where the focusing strength, $k(z)$, is related to the quadrupole gradient according to

$$k(z) = \frac{q}{p^o}g(z). \tag{80}$$

Note that this is sometimes written

$$k(z) = g(z)/B\rho, \tag{81}$$

where $B\rho$ is the so-called magnetic rigidity of the reference particle,

$$B\rho \equiv \frac{p^o}{q}. \tag{82}$$

Now we will restrict ourselves to the linear map, which is governed by H_2. In this case, only the leading order term in A_z is required. (This should be evident from Eq. (49) and Eq. (74), since the leading term in A_z is quadratic, and this is the term that appears outside the square root in Eq. (49); in contrast, A_x and A_y contain only third order and higher order terms, which will result in third order and higher order terms in the Hamiltonian when Eq. (49) is expanded around the given trajectory.) It is easy to show that, for $g =$ constant, the matrix M is given by

$$M = \left(\begin{array}{cc|cc|cc} \cos\sqrt{k}z & \frac{1}{\sqrt{k}}\sin\sqrt{k}z & & & & \\ -\sqrt{k}\sin kz & \cos\sqrt{k}z & & & & \\ \hline & & \cosh\sqrt{k}z & \frac{1}{\sqrt{k}}\sinh\sqrt{k}z & & \\ & & \sqrt{k}\sinh\sqrt{k}z & \cosh\sqrt{k}z & & \\ \hline & & & & 1 & \frac{1}{\gamma_o^2\beta_o^2}z \\ & & & & 0 & 1 \end{array}\right). \tag{83}$$

Symplectic approximation of Quad maps

For a thin magnet of of length $L \Rightarrow 0$, with $1/f = k \times L$ fixed, and ignoring the longitudunal coordinates, the quadrupole map becomes:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{f} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & \frac{1}{f} & 1 \end{pmatrix}. \tag{84}$$

Ignoring x-y coupling, we can study the 2×2 blocks separately. In this case, the symplectic condition is

$$detM = 1, \tag{85}$$

for M any 2×2 block (remember lecture 01). Obviously the components of 84 (and the full matrix itself) satisfy condition 85.

Let us use one block of the quadrupole map 83 to propagate a step Δz:

$$M_{z \Rightarrow z+\Delta z} = \begin{pmatrix} cos\sqrt(k)\Delta z & \frac{1}{\sqrt(k)} sin\sqrt(k)\Delta z \\ -\sqrt(k) sin\sqrt(k)\Delta z & cos\sqrt(k)\Delta z \end{pmatrix}. \tag{86}$$

If we Taylor expand 86, we obtain

$$M_{z \Rightarrow z+\Delta z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \Delta z \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} + \Delta z^2 \begin{pmatrix} -\frac{k}{2} & 0 \\ 0 & -\frac{k}{2} \end{pmatrix} + ... \tag{87}$$

If we truncate the series, keeping up to the first non-trivial term of equation 87 (Δz term), we obtain a map which violates the condition 85. This simple example illustrates the problem faced by matrix codes, which represent each magnet using a Taylor series expansion arround the design trajectory. Although very high orders could be obtained (especially with the use of Lie algebraic methods and automatic differentiation techniques), unless a symplectification procedire is applied, long term tracking will be problematic. In addition, simplectification by adding extra terms is an ad hoc way could also be problematic beyond a certain radius in phase-space (large amplitudes, where they could degrade the accuracy of the Taylor map).

Despite the above discussion, we proceed to add an order Δz^2 term to the first non-trivial term of equation 87, to obtain:

$$M_{z \Rightarrow z+\Delta z} \approx \begin{pmatrix} 1 & \Delta z \\ -k\Delta z & 1 - k\Delta z^2 \end{pmatrix} \tag{88}$$

This matrix differs from the one that we would have obtained by truncating equation 87 to order Δz^2, but it obeys condition 85.

- **Problem:** verify both statements above

- **Problem:** Consider a 1D quadrupole of length L made of drift(s) and one thin quadrupole kick:

  1. Construct all possible maps
  2. Are all the maps of equal in accuracy?
  3. Relate your observation to our earlier discussion on split operator integrators.

- Plot phase-space trajectories using the maps from equations 86, the order Δz truncated map, the order Δz^2 truncated maps, the "symplectified" map from equation 88, and the maps from the previous problem. For simplicity, set k to 1.

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